GENERALIZED EXTENDED BETA FUNCTION

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DOI: https://doi.org/10.56293/IJASR.2022.5544

IJASR 2023
VOLUME 6
ISSUE 4 JULY – AUGUST **ISSN: 2581-7876**

Abstract: Special functions are crucial in defining the concept of fractional calculus. Over the years, numerous extensions and generalizations of the special functions were explored by many researchers. This paper presents a generalization of the extended beta function in [1]. Some integral representations of the generalized extended beta functions, properties and Mellin transforms were also investigated. Moreover, the generalized Gauss, Appel and Lauricella hypergeometric functions, Riemann-Liouville fractional derivative operator and the generalized extended beta distribution were also discussed.

Keywords: Extended beta function, extended gamma function, modified Bessel function, hypergeometric function, Mellin transform and Fractional derivative.

Introduction

The classical gamma and beta functions are defined as

$$\Gamma(r) = \int_0^\infty t^{r-1} e^{-t} dt \tag{1.1}$$

and

$$B(r,s) = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)} = \int_0^1 t^{r-1}(1-t)^{s-1} dt \tag{1.2}$$

Equation (1.1) was first extended by Chaudhry et al [2] in 1994 as

$$\Gamma_p(r) = \int_0^\infty t^{r-1} exp\left(-t-\frac{p}{t}\right) dt, \qquad Re(p) \geq 0, \qquad Re(r) \geq 0 \tag{1.3}$$

And (1.2) to (1.4) by the same authors [2] in 1997 as

$$B_p(r,s) = \int_0^1 t^{r-1}(1-t)^{s-1} exp\left(-\frac{p}{t(1-t)}\right) dt, Re(p) \geq 0 \tag{1.4}$$

In 2011, Lee et al [8] considered the generalisation of the above extended beta function and defined the generalised extended beta function as

$$B(r,s;p;m) = \int_0^1 t^{r-1}(1-t)^{s-1} exp\left(-\frac{p}{t^m(1-t)^m}\right) dt,\ Re(p) > 0, m > 0 \tag{1.5}$$

Ozergin in 2011, extended the beta function using the confluent hypergeometric function (see for example [7]) and defined the new extended beta function as

$$B_p^{(\alpha,\beta)}(r,s) = \int_0^1 t^{r-1}(1-t)^{s-1} {}_1F_1\left(\alpha,\beta;\frac{-P}{t(1-t)}\right) dt, \tag{1.6}$$

where $Re(p) > 0,\quad Re(\alpha) > 0,\quad Re(\beta) > 0$ and

$$ {}_1F_1(\alpha,\beta;z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\beta)_n}\frac{z^n}{n!} \tag{1.7}$$

In the same year, Lee et al [8] generalised the extended beta function given in (1.6) and defined the new generalised extended beta function as

$$ B_p^{(\alpha,\beta;m)}(r,s) = \int_0^1 t^{r-1}(1-t)^{s-1}\ {}_1F_1\left(\alpha,\beta;\frac{-P}{t^m(1-t)^m}\right)dt, \tag{1.8}$$

$Re(p) > 0,\quad min\{Re(r), Re(s), Re(\alpha), Re(\beta)\} > 0$ and $Re(m) > 0$

Putting $m = 1$ in (1.8) reduces to (1.6).

In 2018, Shadab et al [9] introduced an extended beta function using 1 parameter Mittag-Leffler function $E_\alpha\left(-\frac{p}{t(1-t)}\right)$ and defined the new extended beta function as

$$ B_p^{\alpha}(r,s) = \int_0^1 t^{r-1}(1-t)^{s-1}\, E_\alpha\left(-\frac{p}{t(1-t)}\right)dt \tag{1.9}$$

$$ Re(r) > 0, Re(s) > 0 \text{ and } E_\alpha(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)} \tag{1.10}$$

The above extension was generalized by Rahman [4] as

$$ B_p^{(\alpha;m)}(r,s) = \int_0^1 t^{r-1}(1-t)^{s-1}\, E_\alpha\left(-\frac{p}{t^m(1-t)^m}\right)dt \tag{1.11}$$

$Re(r) > 0,\ Re(s) > 0, Re(p) \geq 0$, and $\alpha, m > 0$.

Putting $m = 1$ in (1.11) gives (1.9)

Parmar [6] gives further results on the extended beta function using modified Bessel function of order $n+\frac{1}{2}$ given by Chaudhry in [2] as a relationship between the extended beta function and the MacDonald function and defined

$$ B_n(r,s;p) = \sqrt{\frac{2p}{\pi}}\int_0^1 t^{r-\frac{3}{2}}(1-t)^{s-\frac{3}{2}}K_{n+\frac{1}{2}}\left(\frac{p}{t(1-t)}\right)dt \tag{1.12}$$

$Re(r) > 0,\ Re(s) > 0, Re(p) > 0$ and

$$ K_{n+\frac{1}{2}}(z) = \sqrt{\frac{2p}{\pi}}e^{-z}\sum_{k=0}^{n}\frac{(2z)^{-m}}{m!}\frac{\Gamma(n+m+1)}{\Gamma(n-m+1)} \tag{1.13}$$

A generalized extension of the beta function in [6] was proposed in [10]

$$ B_n(r,s;p;m) = \sqrt{\frac{2p}{\pi}}\int_0^1 t^{r-\frac{3}{2}}(1-t)^{s-\frac{3}{2}}K_{n+\frac{1}{2}}\left(\frac{p}{t^m(1-t)^m}\right)dt \tag{1.14}$$

$Re(r) > 0,\ Re(s) > 0, Re(p) > 0$ and $Re(m) > 0$.

When $m = 1$, equation (1.14) reduces to the extended beta function defined in [6].

The second part of this paper will further generalise the generalised extended beta function in [10].

2. Main Work

We now introduce the new generalised extended beta function as

$$B_n(r,s;p;\gamma;\eta)=\sqrt{\frac{2p}{\pi}}\int_0^1 t^{r-\frac{3}{2}}(1-t)^{s-\frac{3}{2}}K_{n+\frac{1}{2}}\left(\frac{p}{t^{\gamma}(1-t)^{\eta}}\right)dt \quad (2.1)$$

$\mathrm{Re}(r)>0$, $Re(s)>0$, $\mathrm{Re}(p)>0$, $Re(\gamma)>0$, and $Re(\eta)>0$.

Remark:

1) When $\gamma=\eta$, equation (2.1) reduces to the generalised extended beta function defined in [10].

2) For $\gamma=\eta=1$, (2.1) gives the extended beta function in [2, 6].

Integral Representations of the New Generalised Extended Beta Function $B_n(r,s;p;\gamma;\eta)$

Theorem 1: The following integral representations holds true:

$$B_n(r,s;p;\gamma;\eta)=2\sqrt{\frac{2p}{\pi}}\int_0^{\frac{\pi}{2}}cos^{2r-2}\theta\, sin^{2s-2}\theta\, K_{n+\frac{1}{2}}(p\,sec^{2\gamma}\theta\, csc^{2\eta}\theta)d\theta \quad (2.2)$$

$$B_n(r,s;p;\gamma;\eta)=\sqrt{\frac{2p}{\pi}}\int_0^{\infty}\frac{u^{r-\frac{3}{2}}}{(1+u)^{r+s-1}}K_{n+\frac{1}{2}}\left[\frac{p(1+u)^{\gamma+\eta}}{u^{\gamma}}\right]du \quad (2.3)$$

$$B_n(r,s;p;\gamma;\eta)=2^{2-r-s}\sqrt{\frac{2p}{\pi}}\int_0^1(1+u)^{r-\frac{3}{2}}(1-u)^{s-\frac{3}{2}}K_{n+\frac{1}{2}}\left(\frac{2^{\gamma+\eta}p}{(1+u)^{\gamma}(1-u)^{\eta}}\right)du \quad (2.4)$$

Proof

To prove equation (2.2), (2.3) and (2.4), the transformations $t=cos^2\theta$, $t=\frac{u}{u+1}$ and $t=\frac{u+1}{2}$ are to be used respectively.

Theorem 2:

$$B_n(r,s;p;\gamma;\eta)=\sqrt{\frac{2p}{\pi}}\int_0^{\infty}\frac{x^{r-\frac{3}{2}}}{(1+x)^{r+s-1}}K_{n+\frac{1}{2}}\left[\frac{p(1+x)^{\gamma+\eta}}{x^{\eta}}\right]dx \quad (2.5)$$

$$B_n(r,s;p;\gamma;\eta)=\frac{1}{2}\sqrt{\frac{2p}{\pi}}\int_0^{\infty}\frac{x^{r-\frac{3}{2}}+x^{s-\frac{3}{2}}}{(1+x)^{r+s-1}}K_{n+\frac{1}{2}}\left[\frac{p(1+x)^{\gamma+\eta}}{x^{\gamma}}\right]dx \quad (2.6)$$

Proof

Inserting $t=\frac{1}{1+x}$ in (2.1), $dt=\frac{-1}{(1+x)^2}dx$, $t=0: x\to\infty$ and $t=0: x=0$

$$B_n(r,s;p;\gamma;\eta)=\sqrt{\frac{2p}{\pi}}\int_0^{\infty}\frac{x^{s-\frac{3}{2}}}{(1+x)^{r+s-1}}K_{n+\frac{1}{2}}\left[\frac{p(1+x)^{\gamma+\eta}}{x^{\eta}}\right]dx$$

On using the symmetric property of beta function, equation (2.5) is obtained.

From (2.5),

$$B_n(r,s;p;\gamma;\eta) = \sqrt{\frac{2p}{\pi}}\int_0^1 \frac{x^{s-\frac{3}{2}}}{(1+x)^{r+s-1}} K_{n+\frac{1}{2}}\left[\frac{p(1+x)^{\gamma+\eta}}{x^{\eta}}\right]dx + \sqrt{\frac{2p}{\pi}}\int_1^{\infty} \frac{x^{s-\frac{3}{2}}}{(1+x)^{r+s-1}} K_{n+\frac{1}{2}}\left[\frac{p(1+x)^{\gamma+\eta}}{x^{\eta}}\right]dx$$

From the second part of the above equation, put$x = \frac{1}{t}, dx = \frac{-dt}{t^2}$,$x = 1: t = 1$ and $x \to \infty: t = 0$

$$\sqrt{\frac{2p}{\pi}}\int_1^{\infty} \frac{x^{s-\frac{3}{2}}}{(1+x)^{r+s-1}} K_{n+\frac{1}{2}}\left[\frac{p(1+x)^{\gamma+\eta}}{x^{\eta}}\right]dx = \sqrt{\frac{2p}{\pi}}\int_1^{0} \frac{\left(\frac{1}{t}\right)^{s-\frac{3}{2}}}{(1+\frac{1}{t})^{r+s-1}} K_{n+\frac{1}{2}}\left[\frac{p(1+\frac{1}{t})^{\gamma+\eta}}{(\frac{1}{t})^{\eta}}\right]\frac{-dt}{t^2}$$

$$= \sqrt{\frac{2p}{\pi}}\int_0^{1} \frac{t^{s-\frac{3}{2}}}{(1+t)^{r+s-1}} K_{n+\frac{1}{2}}\left[\frac{p(1+t)^{\gamma+\eta}}{t^{\gamma}}\right]dx$$

Putting (2.6) in (2.5), we get the desired result.

Theorem 3 The following integral representations also holds true:

$$B_n(r,s;p;\gamma;\eta) = \lambda^{r-\frac{1}{2}}\mu^{s-\frac{1}{2}}\sqrt{\frac{2p}{\pi}}\int_0^{\infty} \frac{x^{r-\frac{3}{2}}}{(\lambda x+\mu)^{r+s-1}} K_{n+\frac{1}{2}}\left[\frac{p(\mu+\lambda x)^{\gamma+\eta}}{(\lambda x)^{\gamma}\mu^{\eta}}\right]dx \qquad (2.7)$$

$$B_n(r,s;p;\gamma;\eta) = 2\lambda^{r-\frac{1}{2}}\mu^{s-\frac{1}{2}}\sqrt{\frac{2p}{\pi}}\int_0^{\frac{\pi}{2}} \frac{sin^{2s-2}\,\theta\, cos^{2r-2}\,\theta}{(\lambda\, sin^2\theta+\mu\, cos^2\theta)^{r+s-1}} K_{n+\frac{1}{2}}\left(\frac{p(\mu+\lambda\, tan^2\theta)^{\gamma+\eta}}{\lambda^{\gamma}\, tan^{2\gamma}\theta\,\mu^{\eta}}\right)d\theta \quad (2.8)$$

Proof

Putting $x = \frac{\lambda}{\mu}t$ in (2.5),$dx = \frac{\lambda}{\mu}dt$, $x = 0: t = 0$ and $x \to \infty: t \to \infty$

$$B_n(r,s;p;\gamma;\eta) = \sqrt{\frac{2p}{\pi}}\int_0^{\infty} \frac{(\frac{\lambda}{\mu}t)^{r-\frac{3}{2}}}{(1+\frac{\lambda}{\mu}t)^{r+s-1}} K_{n+\frac{1}{2}}\left[\frac{p(1+\frac{\lambda}{\mu}t)^{\gamma+\eta}}{(\frac{\lambda}{\mu}t)^{\gamma}}\right]\frac{\lambda}{\mu}dt$$

On rearranging and simplifying the above equation, we obtain the desired result.

To prove (2.8), put $x = tan^2\theta$ in (2.7),

$$dx = 2\,tan\,\theta\, sec^2\,\theta\, d\theta,\; x = 0: \theta = 0 \text{ and } x \to \infty: \theta = \frac{\pi}{2}$$

$$B_n(r,s;p;\gamma;\eta) = \lambda^{r-\frac{1}{2}}\mu^{s-\frac{1}{2}}\sqrt{\frac{2p}{\pi}}\int_0^{\frac{\pi}{2}} \frac{tan^{2r-3}\,\theta}{(\lambda\, tan^2\theta+\mu)^{r+s-1}} K_{n+\frac{1}{2}}\left[\frac{p(\mu+\lambda\, tan^2\theta)^{\gamma+\eta}}{\lambda^{\gamma}\mu^{\eta}\, tan^{2\gamma}\,\theta}\right]2\,tan\,\theta\, sec^2\,\theta\, d\theta$$

On simplifying the above equation, we obtain the desired result.

Theorem 4:

$$B_n(r,s;p;\gamma;\eta) = \lambda^{s-\frac{1}{2}}\mu^{r-\frac{1}{2}}\sqrt{\frac{2p}{\pi}}\int_0^{1} \frac{x^{r-\frac{3}{2}}(1-x)^{s-\frac{3}{2}}}{[\lambda+(\mu-\lambda)x]^{r+s-1}} K_{n+\frac{1}{2}}\left[\frac{p[\lambda+(\mu-\lambda)x]^{\gamma+\eta}}{(\mu x)^{\gamma}[\lambda(1-x)]^{\eta}}\right]dx \qquad (2.9)$$

$$B_n(r,s;p;\gamma;\eta) = \mu^{s-\frac{1}{2}}(\mu+\tau)^{r-\frac{1}{2}}\sqrt{\frac{2p}{\pi}}\int_0^1 \frac{x^{r-\frac{3}{2}}(1-x)^{s-\frac{3}{2}}}{[\lambda+(\mu-\lambda)x]^{r+s-1}} K_{n+\frac{1}{2}}\left[\frac{p[\mu+(1-x)\sigma]^{\gamma+\eta}}{(\mu x)^{\gamma}(\mu+\sigma)^{\eta}(1-x)^{\eta}}\right] dx \quad (2.10)$$

Proof

(2.9) can be obtained by using the transformation $\frac{\lambda}{x} - \frac{\mu}{t} = \lambda - \mu$ in (2.1).

Inserting $\lambda - \mu = \sigma$in (2.9) gives (2.10).

Theorem 5: The following integral representations hold true:

$$B_n(r,s;p;\gamma;\eta) = \frac{1}{(\lambda-\mu)^{r+s-2}}\sqrt{\frac{2p}{\pi}}\int_0^1 (x-\mu)^{r-\frac{3}{2}}(\lambda-x)^{s-\frac{3}{2}} K_{n+\frac{1}{2}}\left(\frac{p(\lambda-\mu)^{\gamma+\eta}}{(x-\mu)^{\gamma}(\lambda-x)^{\eta}}\right) dx \quad (2.11)$$

$$\int_{-1}^{1}(1+x)^{r-\frac{3}{2}}(1-x)^{s-\frac{3}{2}} K_{n+\frac{1}{2}}\left(\frac{2^{\gamma+\eta}p}{(1+x)^{\gamma}(1-x)^{\eta}}\right) dx = 2^{r+s-2}\sqrt{\frac{\pi}{2p}} B_n(r,s,p;\gamma;\eta) \quad (2.12)$$

Proof

Using the transformation $t = \frac{x-\mu}{\lambda-\mu}$in (2.1), $dt = \frac{dx}{\lambda-\mu}$, $t = 0: x = u, t = 1: x = \lambda$ and

$$B_n(r,s;p;\gamma,\eta) = \sqrt{\frac{2p}{\pi}}\int_{\mu}^{\lambda}\left(\frac{x-\mu}{\lambda-\mu}\right)^{r-\frac{3}{2}}\left(\frac{\lambda-x}{\lambda-\mu}\right)^{s-\frac{3}{2}} K_{n+\frac{1}{2}}\left(\frac{p}{\left(\frac{x-\mu}{\lambda-\mu}\right)^{\gamma}\left(\frac{\lambda-x}{\lambda-\mu}\right)^{\eta}}\right)\cdot\frac{dx}{\lambda-\mu}$$

On simplifying the right-hand side of the above equation, we get the desired result.

Substituting $\lambda = 1$and $\mu = -1$ in (2.11) gives (2.12).

3. The Generalized Extended Hypergeometric Functions

Definition: The generalized extended Gauss hypergeometric function is defined as

$${}_2F_1(\vartheta_1,\vartheta_2,\vartheta_3;t;p;\gamma,\eta) = \sum_{n=0}^{\infty}(\vartheta_1)_n \frac{B_n(\vartheta_2+n,\vartheta_3-\vartheta_2;p;\gamma,\eta)}{B(\vartheta_2,\vartheta_3-\vartheta_2)}\frac{t^n}{n!} \quad (3.1)$$

$Re(p) > 0, Re(n) \geq 0, Re(\vartheta_3) > Re(\vartheta_2) > 0$ and $|t| < 1$.

Definition: The generalized extended Appel hypergeometric functions are defined as

$$F^2(\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_4;t_1,t_2;p;\gamma,\eta) = \sum_{n,m=0}^{\infty}(\vartheta_2)_n(\vartheta_3)_n B_n(\vartheta_1+n+m,\vartheta_4-\vartheta_1;p;\gamma,\eta)\frac{t_1{}^n}{n!}\frac{t_2{}^m}{m!} \quad (3.2)$$

$Re(p) > 0, Re(n) \geq 0, Re(\vartheta_4) > Re(\vartheta_1) > 0$ and $|t_1| < 1, |t_2| < 1$.

and $F^2(\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_4,\vartheta_5;t_1,t_2;p;\gamma,\eta) = \sum_{n,m=0}^{\infty}(\vartheta_1)_{n+m}\frac{B_n(\vartheta_2+n,\vartheta_4-\vartheta_2;p;\gamma,\eta)}{B(\vartheta_2,\vartheta_4-\vartheta_2)}\frac{B_n(\vartheta_3+m,\vartheta_5-\vartheta_3;p;\gamma,\eta)}{B(\vartheta_3,\vartheta_5-\vartheta_3)}\frac{t_1{}^n}{n!}\frac{t_2{}^m}{m!}$ (3.3)

$Re(p) > 0, Re(n) \geq 0, Re(\vartheta_4) > Re(\vartheta_2) > 0, Re(\vartheta_5) > Re(\vartheta_3) > 0$ and $|t_1| + |t_2| < 1$.

Definition: The generalized extended Lauricella hypergeometric function is defined as

$F^2(\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_4,\vartheta_5;t_1,t_2,t_3;p;\gamma,\eta) = \sum_{n,m,k=0}^{\infty}(\vartheta_2)_n(\vartheta_3)_n(\vartheta_4)_n B_n(\vartheta_1+\mu+m+k,\vartheta_5 - \vartheta_1;p;\gamma,\eta)\frac{t_1{}^n}{n!}\frac{t_2{}^m}{m!}\frac{t_3{}^k}{k!}$ (3.4)

$Re(p) > 0, Re(n) \geq 0, Re(\vartheta_5) > Re(\vartheta_1) > 0$ and $|t_1| < 1, |t_2| < 1, |t_3| < 1.$

Theorem 6: The following integral representation holds true.

$$F(\vartheta_1, \vartheta_2, \vartheta_3; t_1; p; \gamma, \eta) = \sqrt{\frac{2p}{\pi}} \frac{1}{B(\vartheta_2, \vartheta_3 - \vartheta_2)} \int_0^1 t^{\vartheta_1 - \frac{3}{2}} (1-t)^{\vartheta_3 - \vartheta_2 - \frac{3}{2}} (1 - t_1 t)^{-\vartheta_1} K_{n+\frac{1}{2}} \left(\frac{p}{t^\gamma (1-t)^\eta} \right) dt$$

(3.5)

$$|arg(1 - t_1)| < \pi, Re(p) > 0, Re(n) \geq 0, \gamma, \eta > 0.$$

Proof

By using the relationship between beta function and hypergeometric function together with the well-known relation

$(1 - t_1 t)^{-\vartheta} = \sum_{n=0}^{\infty} (\vartheta)_n \frac{(t_1 t)^n}{n!}$, we obtain the required result. (3.6)

Theorem 7: The following integral representation holds true.

$$F(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4; t_1, t_2; p; \gamma, \eta) = \sqrt{\frac{2p}{\pi}} \frac{1}{B(\vartheta_1, \vartheta_4 - \vartheta_1)} \int_0^1 t^{\vartheta_1 - \frac{3}{2}} (1-t)^{\vartheta_4 - \vartheta_1 - \frac{3}{2}} (1 - t_1 t)^{-\vartheta_2} (1 - t_2 t)^{-\vartheta_3} K_{n+\frac{1}{2}} \left(\frac{p}{t^\gamma (1-t)^\eta} \right) dt \quad (3.7)$$

Proof

Applying (2.1) in (3.3)**,** we have

$$F(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4; t_1, t_2; p; \gamma, \eta) = \sum_{n,m=0}^{\infty} \sqrt{\frac{2p}{\pi}} \frac{1}{B(\vartheta_1, \vartheta_4 - \vartheta_1)} \int_0^1 t^{\vartheta_1 + n + m - \frac{3}{2}} (1 - t)^{\vartheta_4 - \vartheta_1 - \frac{3}{2}} K_{n+\frac{1}{2}} \left(\frac{p}{t^\gamma (1-t)^\eta} \right) dt (\vartheta_2)_n (\vartheta_3)_m \frac{{t_1}^n}{n!} \frac{{t_2}^m}{m!} \quad (3.8)$$

On simplifying the right-hand side of the above equation, we get the required result.

Lemma: For a bounded sequence $\{f(\delta)\}_{\delta=0}^{\infty}$ of essentially arbitrary complex numbers, we have

$$\sum_{\delta=0}^{\infty} f(\delta) \frac{(t_1 + t_2)^\delta}{\delta!} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} f(t_1 + t_2) \frac{{t_1}^n}{n!} \frac{{t_2}^m}{m!} \quad (3.9)$$

Theorem 8: The following integral representations hold true;

$$F(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5; t_1, t_2; p; \gamma, \eta) = \frac{2p}{\pi} \frac{1}{B(\vartheta_2, \vartheta_4 - \vartheta_2)} \frac{1}{B(\vartheta_3, \vartheta_5 - \vartheta_3)} \int_0^1 \int_0^1 t^{\vartheta_2 - \frac{3}{2}} (1-t)^{\vartheta_4 - \vartheta_2 - \frac{3}{2}} v^{\vartheta_2 - \frac{3}{2}} (1 - v)^{\vartheta_5 - \vartheta_3 - \frac{3}{2}} (1 - t t_1 - v t_2)^{-\vartheta_1} \times K_{n+\frac{1}{2}} \left(\frac{p}{t^\gamma (1-t)^\eta} \right) K_{n+\frac{1}{2}} \left(\frac{p}{v^\gamma (1-v)^\eta} \right) dt dv \quad (3.10)$$

Proof

Putting (2.1) in (3.3), we have

$$F^2(\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_4,\vartheta_5;t_1,t_2;p;\gamma,\eta)=\frac{2p}{\pi}\sum_{n,m=0}^{\infty}\times\left\{\int_0^1 t^{\vartheta_2+n-\frac{3}{2}}(1-t)^{\vartheta_4-\vartheta_2-\frac{3}{2}}K_{n+\frac{1}{2}}\left(\frac{p}{t^{\gamma}(1-t)^{\eta}}\right)dt\right\} \times$$
$$\left\{\int_0^1 v^{\vartheta_2+n-\frac{3}{2}}(1-v)^{\vartheta_5-\vartheta_3-\frac{3}{2}}K_{n+\frac{1}{2}}\left(\frac{p}{v^{\gamma}(1-v)^{\eta}}\right)dv\right\}\frac{(\vartheta_1)_{n+m}}{B(\vartheta_2,\vartheta_4-\vartheta_2)B(\vartheta_3,\vartheta_5-\vartheta_3)}\frac{t_1{}^n}{n!}\frac{t_2{}^m}{m!} \quad (3.11)$$

Interchanging the order of integration and summation in (3.11) gives

$$F^2(\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_4,\vartheta_5;t_1,t_2;p;\gamma,\eta)=\frac{2p}{\pi}\frac{1}{B(\vartheta_2,\vartheta_4-\vartheta_2)B(\vartheta_3,\vartheta_5-\vartheta_3)}$$
$$\times\int_0^1\int_0^1 t^{\vartheta_2-\frac{3}{2}}(1-t)^{\vartheta_4-\vartheta_2-\frac{3}{2}}v^{\vartheta_2-\frac{3}{2}}(1-v)^{\vartheta_5-\vartheta_3-\frac{3}{2}}K_{n+\frac{1}{2}}\left(\frac{p}{t^{\gamma}(1-t)^{\eta}}\right)K_{n+\frac{1}{2}}\left(\frac{p}{v^{\gamma}(1-v)^{\eta}}\right)$$
$$\times\left(\sum_{n,m=0}^{\infty}(\vartheta_1)_{n+m}\frac{(tt_1)^n}{n!}\frac{(vt_2)^m}{m!}\right)dtdv$$

Applying (3.9), we obtain the desired result.

Theorem 9: The following integral representation also holds

$$F^3(\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_4,\vartheta_5;t_1,t_2,t_3;p;\gamma,\eta)=$$
$$\sqrt{\frac{2p}{\pi}}\frac{1}{B(\vartheta_1,\vartheta_5-\vartheta_1)}\int_0^1\frac{t^{\vartheta_1-\frac{3}{2}}(1-t)^{\vartheta_3-\vartheta_1-\frac{3}{2}}}{(1-t_1t)^{\vartheta_2}(1-t_2t)^{\vartheta_3}(1-t_3t)^{\vartheta_4}}K_{n+\frac{1}{2}}\left(\frac{p}{t^{\gamma}(1-t)^{\eta}}\right)dt \quad (3.12)$$

Proof

The proof of (3.12) is similar to that of (3.10)

4. Generalized Extended Riemann-Liouville Fractional Derivative

The classical Riemann-Liouville fractional derivative is defined as [14]

$$D^{\alpha}\varphi(x)=\frac{d^k}{dx^k}\left\{\frac{1}{\Gamma(k-\alpha)}\int_0^x(x-t)^{k-\alpha-1}\varphi(t)dt\right\} \quad (4.1)$$

An extended Riemann-Liouville fractional derivative is given by [10]

$$D^{\alpha}_{n;p;m}\varphi(x)=\frac{d^k}{dx^k}\left\{\frac{1}{\Gamma(k-\alpha)}\sqrt{\frac{2p}{\pi}}\int_0^x(x-t)^{k-\alpha-1}\varphi(t)K_{n+\frac{1}{2}}\left(\frac{pz^{2m}}{t^m(1-t)^m}\right)dt\right\} \quad (4.2)$$

The generalize extended Riemann-Liouville fractional derivative is defined as

$$D^{\alpha}_{n;p;\gamma,\eta}\varphi(x)=\frac{d^k}{dx^k}\left\{\frac{1}{\Gamma(k-\alpha)}\sqrt{\frac{2p}{\pi}}\int_0^x(x-t)^{k-\alpha-1}\varphi(t)K_{n+\frac{1}{2}}\left(\frac{pz^{\gamma+\eta}}{t^{\gamma}(1-t)^{\eta}}\right)dt\right\} \quad (4.3)$$

Remark 4.1

1. For $\gamma=\eta$, the generalized extended Riemann-Liouville fractional operator reduces to (4.2)
2. When $\gamma=\eta=1, n=0$and $p\to 0$, the generalized extended Riemann-Liouville fractional operator reduces to (4.1)

5. Mellin Transform

Theorem 10: The following relation holds:

$$M\{B(r,s,p;\gamma,\eta):p\to j\}=\frac{1}{2^j}\frac{\Gamma(j+n)\Gamma(\frac{j}{2}-\frac{n}{2})}{\Gamma(\frac{j}{2}+\frac{n}{2})}\frac{\Gamma(r+\gamma j+\frac{\gamma-1}{2})\Gamma(s+\eta j+\frac{\eta-1}{2})}{\Gamma(r+s+(\gamma+\eta)j+\frac{1}{2}(\gamma+\eta-2))} \tag{3.13}$$

Proof

The Mellin transform of a function f(x) is given by

$$M\{f(x):x\to j\}=\int_0^\infty x^{j-1}f(x)dx \tag{3.14}$$

Applying the above the definition to (2.1), we have

$$M\{B(r,s;p;\gamma;\eta):p\to j\}=\int_0^\infty p^{j-1}\sqrt{\frac{2p}{\pi}}\int_0^1 t^{r-\frac{3}{2}}(1-t)^{s-\frac{3}{2}}K_{n+\frac{1}{2}}\left(\frac{p}{t^\gamma(1-t)^\eta}\right)dtdp$$

$$=\sqrt{\frac{2}{\pi}}\int_0^1 t^{r-\frac{3}{2}}(1-t)^{s-\frac{3}{2}}\int_0^\infty p^{j+\frac{1}{2}-1}K_{n+\frac{1}{2}}\left(\frac{p}{t^\gamma(1-t)^\eta}\right)dtdp$$

Using the transformation $\varphi=\frac{p}{t^\gamma(1-t)^\eta}$, we have

$$=\sqrt{\frac{2}{\pi}}\int_0^1 t^{r+\gamma(j+\frac{1}{2})-\frac{3}{2}}(1-t)^{s+\eta(j+\frac{1}{2})-\frac{3}{2}}dt\int_0^\infty \varphi^{j+\frac{1}{2}-1}K_{n+\frac{1}{2}}(\varphi)d\varphi$$

$$=\sqrt{\frac{2}{\pi}}\int_0^1 t^{r+\gamma j+\frac{\gamma}{2}-\frac{1}{2}-1}(1-t)^{s+\eta j+\frac{\eta}{2}-\frac{1}{2}-1}dt\int_0^\infty \varphi^{j+\frac{1}{2}-1}K_{n+\frac{1}{2}}(\varphi)d\varphi$$

Applying (1.2), we have

$$\sqrt{\frac{2}{\pi}}\frac{\Gamma(r+\gamma j+\frac{\gamma-1}{2})\Gamma(s+\eta j+\frac{\eta-1}{2})}{\Gamma(r+s+(\gamma+\eta)j+\frac{1}{2}(\gamma+\eta-2))}\int_0^\infty \varphi^{j+\frac{1}{2}-1}K_{n+\frac{1}{2}}(\varphi)d\varphi$$

Using the relation $M\{K_n(z):z\to j\}=2^{j-2}\Gamma(\frac{j}{2}+\frac{n}{2})\Gamma(\frac{j}{2}-\frac{n}{2})$,

The above integral can also be expressed as

$$\int_0^\infty \varphi^{j+\frac{1}{2}-1}K_{n+\frac{1}{2}}(\varphi)d\varphi=2^{j-\frac{3}{2}}\Gamma(\frac{j}{2}+\frac{n}{2}+\frac{1}{2})\Gamma(\frac{j}{2}-\frac{n}{2})$$

Since $\Gamma(n)\Gamma(n+\frac{1}{2})=2^{1-2n}\sqrt{\pi}\Gamma(2n)$,

$$=\frac{2^{j-\frac{3}{2}}2^{1-2(\frac{j}{2}+\frac{n}{2})}\sqrt{\pi}\Gamma 2(\frac{j}{2}+\frac{n}{2})\Gamma(\frac{j}{2}-\frac{n}{2})}{\Gamma(\frac{j}{2}+\frac{n}{2})}$$

$$=2^{-\frac{1}{2}-j}\sqrt{\pi}\frac{\Gamma(j+n)\Gamma(\frac{j}{2}-\frac{n}{2})}{\Gamma(\frac{j}{2}+\frac{n}{2})}$$

Combining the two parts, we have

$$\sqrt{\frac{2}{\pi}}\frac{\Gamma(r+\gamma j+\frac{\gamma-1}{2})\Gamma(s+\eta j+\frac{\eta-1}{2})}{\Gamma(r+s+(\gamma+\eta)j+\frac{1}{2}(\gamma+\eta-2))}\int_0^\infty \varphi^{j+\frac{1}{2}-1}K_{n+\frac{1}{2}}(\varphi)d\varphi$$

=

$$\sqrt{\frac{2}{\pi}}\frac{\Gamma(r+\gamma j+\frac{\gamma-1}{2})\Gamma(s+\eta j+\frac{\eta-1}{2})}{\Gamma(r+s+(\gamma+\eta)j+\frac{1}{2}(\gamma+\eta-2)}2^{-\frac{1}{2}-j}\sqrt{\pi}\frac{\Gamma(j+n)\Gamma(\frac{j}{2}-\frac{n}{2})}{\Gamma(\frac{j}{2}+\frac{n}{2})}$$

On simplifying the above equation, we get the desired result.

6. The Beta Distribution of the New Generalised Beta Function

$$F(t)=\begin{cases}\sqrt{\frac{2p}{\pi}}\frac{1}{B_n(r,s,p;\gamma;\eta)}t^{r-\frac{3}{2}}(1-t)^{s-\frac{3}{2}}K_{n+\frac{1}{2}}\left(\frac{p}{t^{\gamma}(1-t)^{\eta}}\right), 0<t<1\\ 0, elsewhere\end{cases} \quad (4.1)$$

If i is any real number, and $p>0, -\infty<r<\infty, -\infty<s<\infty; \gamma, \eta>0$.

The moment generating function of the distribution

$$M(t)=\sum_{i=0}^{\infty}\frac{t^i}{i!}E(X^i)$$

$$=\sum_{i=0}^{\infty}\frac{B_n(r+i,s;p;\gamma;\eta)}{B_n(r,s;p;\gamma;\eta)}\frac{t^i}{i!} \quad (4.2)$$

For $i=1$,

$$E(X)=\frac{B_n(r+1,s;p;\gamma;\eta)}{B_n(r,s;p;\gamma;\eta)}=\mu \text{ (Mean)} \quad (4.3)$$

$$E(X^2)-[E(X)]^2=\frac{B_n(r+2,s;p;\gamma;\eta)}{B_n(r,s;p;\gamma;\eta)}-\frac{[B_n(r+1,s;p;\gamma;\eta)]^2}{[B_n(r,s;p;\gamma;\eta)]^2}$$

$$=\frac{B(r,s;p;\gamma;\eta)B(r+2,s;p;\gamma;\eta)-B^2(r+1,s;p;\gamma;\eta)}{B^2(r,s;p;\gamma;\eta)}$$

$$=\sigma^2 \text{ (Variance)} \quad (4.4)$$

The cumulative distribution of (3.1) is

$$F(X)=\frac{B_X(r,s;p;\gamma;\eta)}{B(r,s;p;\gamma;\eta)} \quad (4.5)$$

Where $B_X(r,s;p;\gamma;\eta)$ is a new generalised incomplete beta function defined by;

$$B_X(r,s;p;\gamma;\eta)=\sqrt{\frac{2p}{\pi}}\int_0^X t^{r-\frac{3}{2}}(1-t)^{s-\frac{3}{2}}K_{n+\frac{1}{2}}\left(\frac{p}{t^{\gamma}(1-t)^{\eta}}\right)dt \quad (4.6)$$

Conlusion

This research discuss the generalization of the generalized extended beta function

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