Direct Product of Finite Abelian Group

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Abstract: In this project, finite abelian groups with some theoretical and algebraic structures are considered. The order of each group is factorized completely with factor of higher multiplicity where necessary. This unique factorization will allow for a way of building new groups and understanding a given group better. Essentially, it provides a way of relating the given group to the direct products of some of its subgroups. Finally, it also reveals how a group of a finite order is isomorphically related to one of the direct products satisfying certain relatively prime condition.

Illustrative examples are considered with a demonstration of the applicability of the project.

Keywords: Abelian Group, Direct Product, Finite group, Isomorphism, Abstract Algebra, Group theory, Creating Abelian Group, Abelian Group Formation, Abelian Group structure, Finite Abelian Group

I. Introduction

Groups were studied towards the end of the of the eighteenth and at the beginning of thenineteenth century. Yet it was not until relatively late in the nineteenth century that the notion of an abstract group was introduced. Evariste Galois coined the term "group" and established a connection, now known as Galois theory, between the rescent theory of groups and field theory. In abstract algebra, group theory studies the algebraic structures known as groupoids, semigroupoids, monoids and groups.

A group is an ordered pair $(G, *)$ where G is non-φ and $*$ is a binary operation such that thefollowing postulates hold:

i. Closure law: $\forall$ $a, b \in G$, $a * b = c \in G$ exists and it is a unique element of G.

ii. Associative law: $\forall$ $a, b, c \in G$, we have $(a * b) * c = a * (b * c)$.

iii. Existence of identity: $\exists$ $e \in G$ such that, $a * e = e * a = a$ $\forall a \in G$.

iv. Existence of inverse: $\forall a \in G$, $\exists$ a^{-1} such that $a * a^{-1} = a^{-1} * a = e$

Note that If only 1 and 2 are satisfied then we have a semi-group. If only I, 2, and 3 are satisfied then we have a monoid. We only have a group $(G, *)$ iff I, 2, 3, and 4 are satisfied [1]. A group G is said to be finite if the order of G is equal to K where $K \in Z^+$, written as

$$|G| = K.$$

The word abelian is just a normal way of saying a group is commutative. In other words, an abelian group is a group with one extra rule of the group axioms which is commutativity [3], [9], [10]. Abelian groups were named after the Norwegian mathematician; Niels Henrik Abel [1807 1829].

Now, the definition of an abelian group.

An abelian group is a group A satisfying in addition to the following:

$\forall$ $a, b \in A$, $a * b = b * a$. (The commutative law)

There are ways of constructing a new group from some already existing groups, where the direct product of groups is being applied. For instance, if there are two groups A and B, thena new group G = A B can be gotten from both A and B by the simple and straightforward use of the direct product of the two groups [2], [16].
In the context of abelian groups, the direct product is sometimes referred to as the direct sumand it is denoted by G, H [4], [12]. A direct sum plays an important role in the classification of abelian groups. According to the fundamental theorem of finite Abelian group, every finite

Abelian group can be expressed as the sum of cyclic groups [5], [14].

The order of the direct product $G \times H$ is the product of the orders of G and H. i.e., $|G \times H| = |G||H|$ [11]

The following are some elementary properties of direct products;Let G and H be any two groups, then:

(i.) $G \times H$ is abelian if and only if both G and H are abelian.
(ii.) $G \times H$ is isomorphic to $H \times G$.
(iii.) If G and H are both cyclic finite groups and their orders have no common divisor greater than 1, i.e. their orders are relatively prime, then G H is isomorphic to the group whose order is the product of the order of G and order of H [13].

At the end of this research, the concept of a finite abelian group should have been clearlyunderstood and we should be able to obtain abelian groups from a given finite group by usingthe concept of direct product. Also isomorphic groups are shown in practical form from the abelian groups generated in relation to the given group of finite order.

II. Materials and Methods

We consider various theories serving as a guide for building finite abelian groups from a givenfinite group and how to deduce the structure of each abelian group created with proof. Toconsider the structure of finite abelian group, a famous classical theorem called FundamentalTheorem On Finite Abelian Groups is considered. It is a highly satisfying theorem becauseof its decisiveness. The theorem helps to reveal the structure of finite abelian group and also a good tool for attacking any structural problem about finite abelian groups. It has some arithmetic consequences; for instance, one of its by-products is a precise count of how many non-isomorphic abelian groups are present for a given group of finite order.

The Fundamental theorem on finite abelian groups is stated as:

Let G be be a non trivial finite abelian group. Then G is isomorphic to the direct product of finitely many non trivial cyclic groups of prime power order [5].
Lemma:
Let G be an abelian group and let $|G| = mn$ with $\gcd(m, n) = 1$. Let $A = \{x \in G | x^m = e\}$ and let $B = \{x \in G | x^n = e\}$. Then $G \cong A \times B$.

Lagrange's theorem: Lagrange's theorem states that for any finite group G, the order of every subgroup H of G divides the the order of G.

First Sylow's theorem: For every prime factor p with multiplicity n of the order of a finite group G, there exists a Sylow p-subgroup of G, of order p^n [6].

For a prime number p, a p-Sylow subgroup of a group G is a maximal p-subgroup of G, i.e.,a subgroup of G that is a p-group (so that the order of every group element is a power of p)that is not a proper subgroup of any other p-subgroup of G [7], [15].

Chinese remainder theorem (CRT): The Chinese remainder theorem states that if $m, n \in$

$\mathbb{Z}^+$ and the greatest common divisor (gcd) to be given as $\gcd(m, n) = 1$ then $\mathbb{Z}_{mn} \cong \mathbb{Z}_m \times \mathbb{Z}_n$

[8].

III. Results

Case 1: Consider a group of order 36 and find all the abelian groups up to isomorphism. Start by decomposing the order of the group (36) completely:

$$36 = 2.2.3.3 = 2^2.3^2$$

In this case, there are two primes that completely decompose the order of the group (which are 2 and 3). Applying the first Sylow's theorem, then it is clearly seen that there are 2-Sylow subgroup and 3-Sylow subgroup for this group order.

Now, the next thing is to decompose the power of each prime (which is 2) by using the concept of partitioning of integers.

We have:

2=2
2=1+1
Hence, the the Betti number of 2 is 2.[i.e. p(2) = 2]
By the partitioning above, then:

The possibilities of the 2-Sylow subgroups are $Z_2 \times Z_2$, Z_4.
The possibilities of the 3-Sylow subgroups are $Z_3 \times Z_3$, Z_9 as explained for the case 2 Sylow.
It remains to compute the isomorphic classes of the groups of order 36.

$$Z_2 1 \times Z_2 1 \times Z_3 1 \times Z_3 1$$
$$Z_2 1 \times Z_2 1 \times Z_3 2$$
$$Z_2 2 \times Z_3 1 \times Z_3 1$$
$$Z_2 2 \times Z_3 2$$

Therefore, all the abelian groups of order 36 are given by:

1. $Z_2 \times Z_2 \times Z_3 \times Z_3$
2. $Z_4 \times Z_3 \times Z_3$
3. $Z_2 \times Z_2 \times Z_9$
4. $Z_4 \times Z_9$

To determine which of the abelian groups listed above is isomorphic to the group of order 36, the chinese remainder theorem (CRT) is then applied.

Now, by choosing any of the 4 abelian groups above and picking any 2 factors of the chosen one, it can be seen that the one with gcd of any two factors amounting to 1 is the 4th one. Hence, this implies that the abelian group $Z_4 \times Z_9$ is isomorphic (structurally identical) to the group of order 36.

Case 2: In this case, a group whose order is 144 is being considered. We decompose the order of the group (144) completely:

$$144 = 2.2.2.2.3.3 = 2^4.3^2$$

Two primes (2 and 3) completely divide 144. By implication, there are 2-Sylow subgroupand 3-Sylow subgroup of the group of order 144.

Now, to decompose the power of the primes (2 and 4).

By partitioning of integers:

2 = 2
2 = 1 + 1
and
4 = 4
4 = 1 + 3
4 = 2 + 2
4 = 1 + 1 + 2
4 = 1 + 1 + 1 + 1

Hence, the betti number of 2 is 2. [i.e. p(2) = 2]

The betti number of 4 is: 5. [i.e p (4) = 5]

By multiplying the two Betti number (2 and 5), it implies there will be 10 abelian groups oforder 144.

The possibilities of the 2-Sylow subgroup will be: Z_{16}, $Z_2 \times Z_8$, $Z_4 \times Z_4$, $Z_2 \times Z_2 \times Z_4 \times$, $Z_2 \times Z_2 \times Z_2 \times Z_2$.

The possibilities of the 3-Sylow subgroup will be: Z_9, $Z_3 \times Z_3$.

The isomorphic classes for the group of order 144 are computed as:

$$Z_21 \times Z_21 \times Z_21 \times Z_21 \times Z_31 \times Z_31$$
$$Z_21 \times Z_21 \times Z_21 \times Z_21 \times Z_32$$
$$Z_21 \times Z_21 \times Z_22 \times Z_31 \times Z_31$$
$$Z_21 \times Z_21 \times Z_22 \times Z_32$$
$$Z_22 \times Z_22 \times Z_31 \times Z_31$$
$$Z_22 \times Z_22 \times Z_32$$
$$Z_21 \times Z_23 \times Z_31 \times Z_31$$
$$Z_21 \times Z_23 \times Z_32$$
$$Z_24 \times Z_31 \times Z_31$$
$$Z_24 \times Z_32$$

Hence, all the abelian groups of order 144 are:

1. $Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_3 \times Z_3$
2. $Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_9$
3. $Z_2 \times Z_2 \times Z_4 \times Z_3 \times Z_3$
4. $Z_2 \times Z_2 \times Z_4 \times Z_9$
5. $Z_4 \times Z_4 \times Z_3 \times Z_3$
6. $Z_4 \times Z_4 \times Z_9$
7. $Z_2 \times Z_8 \times Z_3 \times Z_3$
8. $Z_2 \times Z_8 \times Z_9$
9. $Z_{16} \times Z_3 \times Z_3$
10. $Z_{16} \times Z_9$

Now, it remains to determine which of the abelian groups is isomorphic to the group of order 144 by using CRT.

By examining all of the 10 abelian groups above, observe that if any of the abelian groups ispicked and the gcd of any two of its factors is computed, then for all the abelian groups above, notice that if the ninth abelian group is considered, the gcd(3, 16) = 1, but gcd(3, 3) = 3 i.e.the CRT fails, continuing in this fashion, it is only the gcd(9, 16) = 1. Hence, it is only the abelian group $Z_{16} \times Z_9$ that is isomorphic (structurally identical) to the group whose order is 144.

Case 3: Let's consider the group whose order is 1800 and find all its abelian group.By decomposition of the order of the group:

$$1800 = 2^3.3^2.5^2$$

The primes that divide 1800 completely are 2,3 and 5. With this, there will be 2-Sylowsubgroup, 3-Sylow subgroup and 5-Sylow subgroup for the group of order 1800.

The power of the primes 2,3,5 are 3,2,2 respectively. It remains to decompose the power ofthe primes by making use of the concept of partitioning of integers.
$2 = 2$
$2 = 1 + 1$
and
$3 = 3$
$3 = 1 + 2$
$3 = 1 + 1 + 1$

Hence, the Betti number of 2 is 2.

The Betti number of 3 is 3.

With this, then the number of abelian groups for a group of order 1800 will be $2 * 2 * 3 = 12$.
The possibilities of the 2-Sylow subgroup will be Z_8, $Z_2 \times Z_4$, and $Z_2 \times Z_2 \times Z_2$.
The possibilities of the 3-Sylow subgroup will be Z_9, $Z_3 \times Z_3$
The possibilities of the 5-Sylow subgroup will be Z_{25}, $Z_5 \times Z_5$

The next thing is to compute the isomorphic classes of the group of order 1800.We have:

$Z_21 \times Z_21 \times Z_21 \times Z_31 \times Z_31 \times Z_51 \times Z_51$
$Z_21 \times Z_21 \times Z_21 \times Z_31 \times Z_31 \times Z_52$
$Z_21 \times Z_21 \times Z_21 \times Z_32 \times Z_51 \times Z_51$
$Z_21 \times Z_21 \times Z_21 \times Z_32 \times Z_52$
$Z_21 \times Z_21 \times Z_21 \times Z_31 \times Z_31 \times Z_51 \times Z_51$
$Z_21 \times Z_22 \times Z_31 \times Z_31 \times Z_51 \times Z_51$
$Z_21 \times Z_22 \times Z_32 \times Z_51 \times Z_51$
$Z_21 \times Z_22 \times Z_32 \times Z_52$
$Z_23 \times Z_31 \times Z_31 \times Z_51 \times Z_51$
$Z_23 \times Z_31 \times Z_31 \times Z_52$
$Z_23 \times Z_32 \times Z_51 \times Z_51$
$Z_23 \times Z_32 \times Z_52$

The abelian groups are:

1. $Z_2 \times Z_2 \times Z_2 \times Z_3 \times Z_3 \times Z_5 \times Z_5$
2. $Z_2 \times Z_2 \times Z_2 \times Z_3 \times Z_3 \times Z_{25}$
3. $Z_2 \times Z_2 \times Z_2 \times Z_9 \times Z_5 \times Z_5$
4. $Z_2 \times Z_2 \times Z_2 \times Z_9 \times Z_{25}$
5. $Z_2 \times Z_4 \times Z_3 \times Z_3 \times Z_5 \times Z_5$
6. $Z_2 \times Z_4 \times Z_3 \times Z_3 \times Z_{25}$
7. $Z_2 \times Z_4 \times Z_9 \times Z_5 \times Z_5$
8. $Z_2 \times Z_4 \times Z_9 \times Z_{25}$
9. $Z_8 \times Z_3 \times Z_3 \times Z_5 \times Z_5$
10. $Z_8 \times Z_3 \times Z_3 \times Z_{25}$
11. $Z_8 \times Z_9 \times Z_5 \times Z_5$
12. $Z_8 \times Z_9 \times Z_{25}$

Now, to determine which of the abelian groups is isomorphic to the given group of order 1800 by applying the CRT.

Picking any of the 12 abelian groups in succession and examining any two of its factors. Consider the abelian group $Z_2 \times Z_4 \times Z_9 \times Z_{25}$, by examining any two of its factors with the CRT, then gcd(9, 25) = 1, gcd(2, 25) = 1, gcd(4, 25) = 1, gcd(2, 9) = 1, gcd(4, 9) = 1, gcd(2, 4) = 2, i.e., the CRT fails here and this implies the abelian group Z_2 Z_2 ×

$Z_4 \times Z_9 \times Z_{25}$ cannot be isomorphic to the group of order 1800, continuing in this fashion and examining the abelian group $Z_8 \times Z_9 \times Z_{25}$, gcd(8, 9) = 1, gcd(8, 25) = 1, and gcd(9, 25) = 1. Hence,it is only the abelian group $Z_8 \times Z_9 \times Z_{25}$ that is isomorphic (structurally identical) to the group of order 1800.

Case 4: For this case consider a group with order 2025.

Start by factorizing the order of the group (2025) completely to get the primes that divideit.

$$2025 = 3^4.5^2$$

By completely decomposing 2025, the primes 3 and 5 divide it. The power of the primes 3 and 5 are 4 and 2 respectively.

Since the primes that divide 2025 are 3 and 5, then; there will be 3-Sylow subgroup and 5-Sylow subgroup for the group of order 2025.

The next thing is to decompose each of this power by using partitioning of integers and thereby getting their Betti number. We have:

2 = 2
2 = 1 + 1
and
4 = 4
4 = 1 + 3
4 = 2 + 2
4 = 1 + 1 + 2
4 = 1 + 1 + 1 + 1

Hence, the the Betti number of of 2 is 2 and the Betti number of 4 is 5. By multiplying their Betti number 2 * 5 = 10, this implies there will be 10 abelian groups of order 2025.

The possibilities of the 3-Sylow subgroup will be: Z_{81},$Z_3 \times Z_{27}$, $Z_3 \times Z_3 \times Z_9$, $Z_9 \times Z_9$ and $Z_3 \times Z_3 \times Z_3 \times Z_3$.

The possibilities of the 5-Sylow subgroup will be: Z_{25} and $Z_5 \times Z_5$.

Now, by the combination of the 3-Sylow subgroup and the 5-Sylow subgroup, the isomorphicclasses will be given as:

$Z_31 \times Z_31 \times Z_31 \times Z_31 \times Z_51 \times Z_51 \times Z_31 \times Z_31 \times Z_31 \times Z_31 \times Z_52$
$Z_31 \times Z_31 \times Z_32 \times Z_51 \times Z_51$
$Z_31 \times Z_31 \times Z_32 \times Z_52$
$Z_31 \times Z_33 \times Z_51 \times Z_51$
$Z_31 \times Z_33 \times Z_52$
$Z_32 \times Z_32 \times Z_51 \times Z_51$
$Z_32 \times Z_32 \times Z_52$
$Z_34 \times Z_51 \times Z_51$
$Z_34 \times Z_52$

By doing a simple computation and getting the powers of the bases, all the abelian groupsof order 2025 will be:

1. $Z_3 \times Z_3 \times Z_3 \times Z_3 \times Z_5 \times Z_5$
2. $Z_3 \times Z_3 \times Z_3 \times Z_3 \times Z_{25}$
3. $Z_3 \times Z_3 \times Z_9 \times Z_5 \times Z_5$
4. $Z_3 \times Z_3 \times Z_9 \times Z_{25}$
5. $Z_3 \times Z_{27} \times Z_5 \times Z_5$
6. $Z_3 \times Z_{27} \times Z_{25}$

7. $Z_9 \times Z_9 \times Z_5 \times Z_5$
8. $Z_9 \times Z_9 \times Z_{25}$
9. $Z_{81} \times Z_5 \times Z_5$
10. $Z_{81} \times Z_{25}$

By getting all the 10 different abelian groups of order 2025, it is now left to determine which of the abelian groups is isomorphic to the group of order 2025. To do this, impose the CRT on each of the abelian groups and take any two factors of any of the abelian groups simultaneously. Taking the abelian group $Z_3 \times Z_{27} \times Z_{25}$; then applying CRT, gcd(3, 25) = 1, gcd(25, 27) = 1 and gcd(3, 27) = 3, which implies that the CRT fails here, therefore theabelian group $Z_3 \times Z_{27} \times Z_{25}$ is not isomorphic to the group of order 2025. Continuing in this fashion, all the other abelian groups also fail the CRT test except for the abelian group $Z_{81} \times Z_{25}$; since the gcd(25, 81) = 1. Hence, it is only the abelian group Z_{81} Z_{25} that is isomorphic (structurally identical) to the group of order 2025.

IV. Conclusion

In this study, different groups of finite orders have been examined such as the groups of orders 36, 144, 1800 and 2025.

The various abelian groups for each group were obtained from the direct product of the possible Sylow's subgroup in each group and the abelian group which is isomorphic (i.e., structurally identical) to the original given group was obtained with the chinese remainder theorem (CRT). These examples demonstrate the applicability of the theorem and corollaries presented in chapter 4. It exhibits the fact that abelian groups of order 814 and above can be used to manufacture isomorphic groups of the same order where one of them is abelian and isomorphic to the original given group.

V. Recommendation

Direct product is a way of producing new groups from an existing group of finite order. It also gives new information about new groups whose algebraic structures can be easily

Determined In a way, this theory enlarges the classes of examples of groups for advancedstudies in abstract algebra and combinatorics.

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